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Abstract. The dynamics of weakly nonlinear wave trains in unstable media is studied. This dynamics is investigated in the framework of a broad class of dynamical systems having a Hamiltonian structure. Two different types of instability are considered. The first one is the instability in a weakly supercritical media. The simplest example of instability of this type is the Kelvin-Helmholtz instability. The second one is the instability due to a weak linear coupling of modes of different nature. The simplest example of a geophysical system where the instability of this and only of this type takes place is the three-layer model of a stratified shear flow with a continuous velocity profile. For both types of instability we obtain nonlinear evolution equations describing the dynamics of wave trains having an unstable spectral interval of wavenumbers. The transformation to appropriate canonical variables turns out to be different for each case, and equations we obtained are different for the two types of instability we considered. Also obtained are evolution equations governing the dynamics of wave trains in weakly subcritical media and in media where modes are coupled in a stable way. Presented results do not depend on a specific physical nature of a medium and refer to a broad class of dynamical systems having the Hamiltonian structure of a special form.

1 Introduction

The methods of Hamiltonian formalism are extremely fruitful for investigating wave field dynamics in stable media. By using the Hamiltonian approach Zakharov (1974) investigated the processes of evolution and interaction of wave trains in dispersive media for a broad class of physical systems. He considered these processes from a general point of view, no matter what the specific nature of a medium is. The principal idea of the theory he developed is a transformation from physical variables of a problem to normal canonical variables. This transformation is related to a fundamental system of eigenvectors of the corresponding linearised problem. For description of weakly nonlinear dynamics and resonant interaction of wave trains one should calculate first several coefficients in the expansion of the Hamiltonian in powers of normal variables. Ignatov (1984) and Goncharov and Pavlov (1993) applied this Hamiltonian approach for investigation of waves in instable media. Romanova (1994) developed Hamiltonian methods for investigation of waves in the region of marginal stability. She considered the weakly nonlinear wave dynamics in the framework of a broad class of dynamical systems subject to some constraints. She showed that the transformation to normal variables is improper in the region of marginal stability, for it leads to a breakdown of the accepted approximation of weak nonlinearity. It is due to normalization of eigenvectors by the quantity $\sqrt{\dfrac{\partial D}{\partial \omega}}$, where $D(\omega, k)$ is the left-hand side of the dispersion equation, and ω is the eigenfrequency related to a certain mode, and this quantity tends to zero in a vicinity of the points where the two eigenvalues coalesce. The paper by Romanova (1994) suggest the way to introduce appropriate canonical variables in the region of marginal stability, where different roots of the dispersion equation are close or equal. Based on these variables the evolution equation for a wave-packet of marginally unstable waves is obtained. But consideration of marginally unstable wave-packets is of purely theoretical interest, since they are usually masked by more rapidly growing modes with wavenumbers in the linearly unstable range. Of greater interest is the consideration of unstable wave trains in weakly unstable media, when the range of wavenumbers is narrow, and the growth-rates of these unstable modes are small. In this case we can investigate the wave dynamics within the weakly nonlinear framework.

The aim of our investigation is the derivation of evolution equations for weakly unstable wave trains in weakly unstable media. We assume that wave-packets are narrow and belong to the range of linear unstable wavenumbers. For the derivation of the evolution equations it is important to distinguish between two types of linear instability. The first type is the instability caused by weak supercriticality of a medium. In this case a small parameter depending on characteristic parameters of a medium enters into equations for wave perturbations. This parameter can be positive as well as negative. The basic state of a medium is unstable for negative values of it, and stable for positive ones. At the critical point of instability when this parameter is equal to zero, we have algebraic instability, i.e. the perturbations grow linearly with time.

The other type of a weak instability occurs due to a weak coupling of modes with different energy signs. In this case a small parameter specifying weak coupling of modes enters into equations. If this parameter is equal to zero, we have two intersecting modes, and no algebraic instability takes place. The consideration below will be carried out for two well-known geophysical models exhibiting these types of instability, although, obviously, it can be generalized to other cases as well. The example of instability of the first type is the Kelvin-Helmholtz instability of a plane vortex sheet. Consider the basic flow of incompressible inviscid fluids in two horizontal parallel infinite streams of different velocities and densities, one stream above the other:

$$U = \begin{cases} u_2, \ z > 0, \\ u_1, \ z < 0, \end{cases} \quad \rho = \begin{cases} \rho_2, \ z > 0, \\ \rho_1, \ z < 0, \end{cases}$$

where $u_{1,2}$ are flow velocities, and $\rho_{1,2}$ are densities in the lower and upper streams respectively, and z is the vertical coordinate. Here and in what follows we assume the flow disturbances to be two-dimensional. The horizontal coordinate along the flow will be denoted by x. It is well-known that if the surface tension σ is taken into account, the condition for instability of the basic flow is

$$\frac{u_1 - u_2}{2} > V_c, \ V_c^2 = \frac{1}{2}\left(\frac{\rho_1 + \rho_2}{\rho_1 \rho_2}\right)\sqrt{g(\rho_1 - \rho_2)\sigma}. \quad (1)$$

If (1) does not hold, the waves are neutrally stable. If the value of $|u_1 - u_2|$ only slightly exceeds the critical value $2V_c$ (weakly supercritical regime), the growth rate of this instability is small, and the interval of instability in the k-axis is narrow.

The second example is the three-layer model of an inviscid stably stratified shear flow considered in detail by Gossard and Hooke (1975), Goncharov (1986), and Romanova (1996). The profiles of the unperturbed flow are

$$\rho = \begin{cases} \rho_3, \ z > h, \\ \rho_2, \ |z| < h, \\ \rho_1, \ z < -h, \end{cases} \quad U = \begin{cases} u, \ z > h, \\ uz/h, \ |z| < h, \\ -u, \ z < -h, \end{cases} \quad (2)$$

where $\rho_3 < \rho_2 < \rho_1$, which means that the density stratification is stable. The velocity profile is continuous in this model but the vorticity experiences jumps at the interfaces. The instability in this model is not of the Kelvin-Helmholtz type. As will be explained later, it occurs owing to the coupling of two stable modes with different energy signs (see Ostrovskiy et al. (1986), Whitham (1974)), Ignatov (1984). As one can see in Figure 38.4 by Gossard and Hooke (1975), the region of instability is narrow, and the growth rates of perturbations are small, if the model parameters having the sense of the Richardson number are large enough. We show that the dispersion equation in this narrow region of wavenumbers k has approximately the same form as that for the K-H weakly unstable waves. However the evolution equations describing the weakly nonlinear dynamics of a wave-packet with the wavenumber spectrum comprising the narrow interval of instability are different in those two cases. The matrix which determines solutions of a linearised system has the form of the Jordan box in the first case and is close to a diagonal one with equal eigenvalues in the second one. It leads to the different canonical structure of equations for these two cases.

The rest of the paper is organised as follows. In Section 2 the nonlinear Klein-Gordon equation is derived on the basis of the developed Hamiltonian approach. This equation governs the evolution of weakly nonlinear and weakly unstable supercritical wave-packets in sub- or supercritical media of K-H type. Section 3 contains the derivation of equations governing the nonlinear wave-packets in the region of weak coupling between modes with the same of opposite energy signs. Section 4 is concerns with the algebraic instability in the media of the K-H type.

2 The derivation of the nonlinear Klein-Gordon equation describing the dynamics of a weakly nonlinear wave-packet in a weakly sub(super) critical Kelvin-Helmholtz flow

The dynamical equation for nonlinear wave-packets in the Kelvin-Helmholtz model was first obtained by Weissman (1979). Here we present a new Hamiltonian approach to the derivation of evolution equations for the wave-packets in the region of instability of the K-H type. Clearly, all the results are valid not only for the K-H instability, but for all the systems where the weak instability takes place for a single mode.

Benjamin and Bridges (1997) have shown that the system of evolution equations describing the Kelvin-Helmholtz instability has the following Hamiltonian form:

$$\eta_t = \frac{\delta H}{\delta \Phi(t,x)}, \ \Phi_t = -\frac{\delta H}{\delta \eta(t,x)}, \quad (3)$$

where $\eta(t,x)$ is the height of the disturbed interface, which we assume to be a single-valued function of the horizontal coordinate x, and $\Phi(t,x) = (\rho_1\Psi_1 - \rho_2\Psi_2)|_{z=\eta}$, where $\rho_{1,2}$ and $\Psi_{1,2}$ are density and velocity potential in the lower and upper layer respectively. In terms of the Fourier-transform system (3) is written in the form:

$$\dot{\eta}(k,t) = \frac{\delta H}{\delta\Phi^*(k,t)},\ \dot{\Phi}(k,t) = -\frac{\delta H}{\delta\eta^*(k,t)}, \tag{4}$$

where the dot indicates the time derivative. The dynamical system (4) can be rewritten in the form considered by Romanova (1994):

$$J(k)\dot{\mathbf{y}}(k) = -\frac{\delta H}{\delta\mathbf{y}(-k)}, \tag{5}$$

where $\mathbf{y}(k,t) = (\Phi(k,t),\ \eta(k,t))$. The condition that the initial variables $\eta(t,x)$ and $\Phi(t,x)$ are real can be written as $\mathbf{y}^*(k,t) = \mathbf{y}(-k,t)$.

The matrix $J(k)$ has the canonical form:

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

It is the particular case of the general class of the Hamiltonian systems considered by Romanova (1994) with the structure matrices $J(k)$ subject to the following conditions:

$$J^*(k) = J(-k),\ J^*(k) = -J'(k) \tag{6}$$

The first term in the expansion of the Hamiltonian H with respect to dependent variables has the following form:

$$\begin{aligned} H_2 &= \tfrac{1}{2}\int\{A(k)\Phi(k)\Phi^*(k) + 2iB(k)\Phi(k)\eta^*(k) \\ &\quad + C(k)\eta(k)\eta^*(k)\}dk. \end{aligned} \tag{7}$$

Here and in what follows the region of integration is the entire real axis. The quantities $A(k)$, $C(k)$ and $B(k)$ are real and are subject to the following conditions:

$$A(k) = A(-k),\ C(k) = C(-k),\ B(k) = -B(-k).$$

An alternative form of equation (7) is

$$H_2 = \frac{1}{2}\int(\mathbf{y}(-k,t), \hat{h}(k)\mathbf{y}(k,t)dk, \tag{8}$$

where

$$\hat{h} = \begin{pmatrix} A(k) & -iB(k) \\ iB(k) & C(k) \end{pmatrix}.$$

The dispersion equation related to the linearised problem is

$$\det\,(\hat{h} - i\omega J) = 0, \tag{9}$$

and the solutions of this equation are

$$\omega_{1,2} = B(k) \pm \sqrt{A(k)C(k)}. \tag{10}$$

In the case of the Kelvin-Helmholtz model the coefficients $A(k), B(k)$ and $C(k)$ are

$$B(k) = k\frac{\rho_1 u_1 + \rho_2 u_2}{\rho_1 + \rho_2},\ A(k) = \frac{|k|}{\rho_1 + \rho_2},$$

$$C(k) = -\frac{\rho_1\rho_2(u_1 - u_2)^2}{\rho_1 + \rho_2}|k| + g\Delta\rho + \sigma k^2,$$

where $\Delta\rho = \rho_1 - \rho_2$. The eigenfrequencies for the K-H linear problem are

$$\omega_{1,2} = k\frac{\rho_1 u_1 + \rho_2 u_2}{\rho_1 + \rho_2} \pm \delta,$$

$$\delta = \sqrt{\left(g\Delta\rho + \sigma k^2 - \frac{\rho_1\rho_2(u_1 - u_2)^2}{\rho_1 + \rho_2}|k|\right)\frac{|k|}{\rho_1 + \rho_2}}. \tag{11}$$

In what follows we use the Boussinesq approximation to simplify calculation. Under this approximation we have the following expressions for the eigenfrequencies

$$\omega_{1,2} = k\frac{u_1 + u_2}{2} \pm \sqrt{g\frac{\Delta\rho}{2\rho}K + \frac{\sigma}{2\rho}K^3 - V^2K^2}, \tag{12}$$

where

$$K = |k|,\ \Delta\rho = \rho_1 - \rho_2,$$

$$V = \frac{u_1 - u_2}{2},\ \rho = \frac{\rho_1 + \rho_2}{2}.$$

The eigenfrequencies $\omega_{1,2}$ are complex if the radicand is negative. Let us consider expression (12) for the frequency ω at the point k_0 where the radicand is equal to zero. This is the critical point of instability. The point k_0 is equal to $\sqrt{g\Delta\rho/\sigma}$, and the critical value of the shear flow velocity V_c is defined as $V_c^2 = \sqrt{g\sigma\Delta\rho/\rho^2}$. Let us introduce a small deviation $\Delta V = V - V_c$ of a shear flow velocity from its critical value V_c. We consider the region of wavenumbers close to wavenumber k_0. Let $k = k_0 + \kappa$. Then for small values of κ/k_0 and $\Delta V/V_c$ the frequency ω in a vicinity of the wavenumber κ_0 is given by the following expression:

$$\omega_{1,2} = \omega_0 \pm \operatorname{sign} k\delta,\ \omega_0 = \hat{\omega}_0 + v\kappa, \tag{13}$$

where

$$v = \frac{u_1 + u_2}{2},\ \delta = \sqrt{\frac{1}{2}V_c^2\kappa^2 - 2V_c\Delta V k_0^2},\ \hat{\omega}_0 = vk_0.$$

If the flow is weakly subcritical, then $\Delta V < 0$, and the eigenfrequencies are real. If it is weakly supercritical, $\omega_{1,2}$ are complex conjugates in a vicinity of the critical wavenumber k_0 and the K-H instability occurs. The growth-rate of this instability is small, and the region of instability in the k-axis is narrow. The following results concern the general systems described by equations of the form (5) and having retained the first term of the

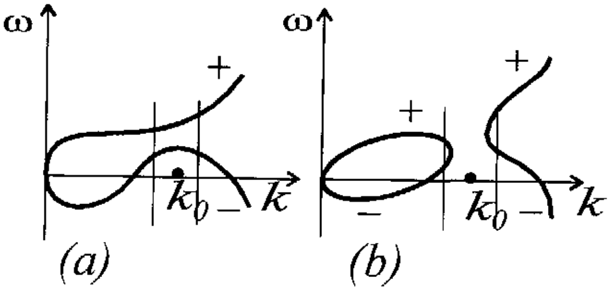

Fig. 1. Dispersion curve for the Kelvin-Helmholtz flow: a) a weakly subcritical case; b) a weakly supercritical case, exponential instability

Hamiltonian expansion in the form (8). Returning to the dispersion equation (10), we assume that the radicand turns to zero at a certain point k_0 and in a vicinity of this point the dispersion relation has the form (13), where

$$\delta = \sqrt{A(k_0+\kappa)C(k_0+\kappa)} \approx \sqrt{V_0^2(\kappa^2 - bk_0^2)}. \tag{14}$$

Here $\kappa = k - k_0$, and b is a small intrinsic dimensionless parameter of a wave system which can be varied. Positive values of b refer to the unstable case, and negative values to the stable one. For the K-H problem this parameter is equal to $b = 4\Delta V/V_c$, i.e. it is proportional to the deviation of a shear velocity from its critical value. The two typical forms of the dispersion curve for the K-H problem are shown schematically in Figure 1. The first one corresponds to a weakly subcritical, and the second one to a weakly supercritical flow.

Following the approach suggested by Romanova (1994), we introduce the new variables $a(k,t)$ by using the following transformation:

$$\mathbf{y}(k,t) = \mathcal{Z}(k)a(k,t) + \mathcal{Z}^*(-k)a^*(-k,t), \tag{15}$$

where the vector $\mathcal{Z}(k)$ is defined as

$$\mathcal{Z}(k) = \begin{cases} \mathcal{Z}_e, & k>0, \\ \mathcal{Z}_a, & k<0, \end{cases}$$

and

$$\mathcal{Z}_e = \frac{\mathbf{z}_1+\mathbf{z}_2}{2}, \; \mathcal{Z}_a = \alpha\frac{\mathbf{z}_1-\mathbf{z}_2}{2}, \tag{16}$$

where $\alpha = \dfrac{2}{\omega_1-\omega_2}$. Here $\omega_{1,2}$ are eigenfrequencies that are determined from the linearised problem (3) and $\mathbf{z}_{1,2}$ are eigenvectors related to these eigenfrequencies and determined from the following system of linear algebraic equations:

$$(\hat{h} - i\omega_{1,2}J)\mathbf{z}_{1,2} = 0. \tag{17}$$

We are going to consider wave-packets with a narrow interval of wavenumbers centred on the point k_0. We assume the flow to be slightly sub or supercritical. The eigenfrequencies are governed by equation (13), where δ is defined by equation (14). One can easily see that $\omega_1 - \omega_2 = 2\operatorname{sign} k\delta$. Eigenvectors of the linearised problem are

$$\mathbf{z}_{1,2} = ((\pm i\delta \operatorname{sign} k/A)\, c_{1,2},\; c_{1,2}),$$

where $c_{1,2}$ are arbitrary constants. We assume that the eigenfrequencies are close to each other in a vicinity of the point k_0, i.e. the radicand $A(k_0)C(k_0) = 0$ at the central point. We use the following properties of eigenfrequencies and eigenvectors: in the stable case, when $\omega_{1,2}$ are real,

$$\omega_{1,2}(-k) = -\omega_{1,2}(k),\; \mathbf{z}^*_{1,2}(-k) = \mathbf{z}_{1,2}(k),$$

and

$$\begin{aligned} (\mathbf{z}_1^*, J\mathbf{z}_2) &= (\mathbf{z}_2^*, J\mathbf{z}_1) = 0, \\ (\mathbf{z}_1^*, J\mathbf{z}_1) &= \frac{2i\delta \text{ sign } k}{A}c_1c_1^*, \\ (\mathbf{z}_2^*, J\mathbf{z}_2) &= -\frac{2i\delta \text{ sign } k}{A}c_2c_2^*; \end{aligned} \tag{18}$$

in the unstable case when ω_1 and ω_2 are complex conjugated,

$$\omega_1^*(-k) = -\omega_2(k),\; \mathbf{z}_1^*(-k) = \mathbf{z}_2(k),$$

and

$$\begin{aligned} (\mathbf{z}_1^*, J\mathbf{z}_1) &= (\mathbf{z}_2^*, J\mathbf{z}_2) = 0, \\ (\mathbf{z}_1^*, J\mathbf{z}_2) &= -\frac{2i\delta\text{sign } k}{A}c_1^*c_2, \\ (\mathbf{z}_2^*, J\mathbf{z}_1) &= -\frac{2i\delta\text{sign } k}{A}c_2^*c_1. \end{aligned} \tag{19}$$

On substituting the transformation (15) into (5) we obtain the following system:

$$\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} \dot{a}(k,t) \\ \dot{a}^*(-k,t) \end{pmatrix} = -\begin{pmatrix} \dfrac{\delta H}{\delta a(-k)} \\ \dfrac{\delta H}{\delta a^*(k)} \end{pmatrix}, \tag{20}$$

where the coefficients b_{ij} of the structure matrix are equal to

$$\begin{aligned} b_{11}(k) &= (\mathcal{Z}(-k), J\mathcal{Z}(k)), \\ b_{12}(k) &= (\mathcal{Z}(-k), J\mathcal{Z}^*(-k)), \\ b_{21}(k) &= (\mathcal{Z}^*(k), J\mathcal{Z}(k)), \\ b_{22}(k) &= (\mathcal{Z}^*(k), J\mathcal{Z}^*(-k)), \end{aligned} \tag{21}$$

or, otherwise,

$$b_{11}(k)=\begin{cases}-\dfrac{\alpha^*}{4}((\mathbf{z}_1^*-\mathbf{z}_2^*),J(\mathbf{z}_1+\mathbf{z}_2)),\ k>0,\\ \dfrac{\alpha}{4}((\mathbf{z}_1^*+\mathbf{z}_2^*),J(\mathbf{z}_1-\mathbf{z}_2)),\ k<0,\end{cases}$$

$$b_{12}(k)=\begin{cases}\dfrac{\alpha^*\alpha}{4}((\mathbf{z}_1^*-\mathbf{z}_2^*),J(\mathbf{z}_1-\mathbf{z}_2)),\ k>0,\\ \dfrac{1}{4}((\mathbf{z}_1^*+\mathbf{z}_2^*),J(\mathbf{z}_1+\mathbf{z}_2)),\ k<0,\end{cases}\tag{22}$$

$$b_{21}(k)=\begin{cases}\dfrac{1}{4}((\mathbf{z}_1^*+\mathbf{z}_2^*),J(\mathbf{z}_1+\mathbf{z}_2)),\ k>0,\\ \dfrac{\alpha^*\alpha}{4}((\mathbf{z}_1^*-\mathbf{z}_2^*),J(\mathbf{z}_1-\mathbf{z}_2)),\ k<0,\end{cases}$$

$$b_{22}(k)=\begin{cases}-\dfrac{\alpha}{4}((\mathbf{z}_1^*+\mathbf{z}_2^*),J(\mathbf{z}_1-\mathbf{z}_2)),\ k>0,\\ \dfrac{\alpha^*}{4}((\mathbf{z}_1^*-\mathbf{z}_2^*),J(\mathbf{z}_1+\mathbf{z}_2)),\ k<0.\end{cases}$$

Using the properties of eigenvectors (18), (19), and assuming the arbitrary constants to be $c_1=c_2^*=i\sqrt{A}$, we obtain that the coefficients of the structure matrix are

$$b_{11}=b_{22}=0,\ b_{12}=b_{21}=-i\mathrm{sign}\,k,\tag{23}$$

for both the stable and unstable regions. Here we assumed that the quantity A is positive, as it takes place in the K-H model. If A is negative, the coefficients b_{12}, b_{21} would change sign. It follows from (23) that system (20) can be written in the form:

$$\dot a(k)=-i\mathrm{sign}\ k\frac{\delta H}{\delta a(-k)}.\tag{24}$$

This equation was obtained by Goncharov and Pavlov (1993) for unstable waves and by Romanova (1994) for marginally unstable ones.

The first term of the expansion of Hamiltonian H in powers of dependent variables $a(k)$ is

$$H_2=\frac{1}{2}\int\left(\begin{pmatrix}a(-k)\\ a^*(k)\end{pmatrix},\hat h_{\rm tr}\begin{pmatrix}a(k)\\ a^*(-k)\end{pmatrix}\right)dk.\tag{25}$$

Our next step is the calculation of components $h_{i,j}$ of the transformed matrix $\hat h_{\rm tr}$. By using equations (17) we obtain the following equations:

$$\hat h\frac{\mathbf{z}_1+\mathbf{z}_2}{2}-iJ\left[\omega_0\left(\frac{\mathbf{z}_1+\mathbf{z}_2}{2}\right)+\delta\left(\frac{\mathbf{z}_1-\mathbf{z}_2}{2}\right)\right]=0,$$

$$\hat h\frac{\mathbf{z}_1-\mathbf{z}_2}{2}-iJ\left[\delta\left(\frac{\mathbf{z}_1+\mathbf{z}_2}{2}\right)+\omega_0\left(\frac{\mathbf{z}_1-\mathbf{z}_2}{2}\right)\right]=0,$$

where $\omega_0=\dfrac{\omega_1+\omega_2}{2}$ and $\delta=\dfrac{\omega_1-\omega_2}{2}$. Turning to new variables (16), we obtain

$$\begin{aligned}&\hat h\mathcal{Z}_e-iJ(\omega_0\mathcal{Z}_e+\delta^2\mathcal{Z}_a)=0,\\ &\hat h\mathcal{Z}_a-iJ(\omega_0\mathcal{Z}_a+\mathcal{Z}_e)=0.\end{aligned}\tag{26}$$

It follows from equations (26) that

$$\begin{aligned}&(\mathcal{Z}_e^*,\hat h\mathcal{Z}_e)=i\omega_0(\mathcal{Z}_e^*,J\mathcal{Z}_e)+i\delta^2(\mathcal{Z}_e^*,J\mathcal{Z}_a),\\ &(\mathcal{Z}_e^*,\hat h\mathcal{Z}_a)=i\omega_0(\mathcal{Z}_e^*,J\mathcal{Z}_a)+i(\mathcal{Z}_e^*,J\mathcal{Z}_e),\\ &(\mathcal{Z}_a^*,\hat h\mathcal{Z}_e)=i\omega_0(\mathcal{Z}_a^*,J\mathcal{Z}_e)+i\delta^2(\mathcal{Z}_a^*,J\mathcal{Z}_a),\\ &(\mathcal{Z}_a^*,\hat h\mathcal{Z}_a)=i\omega_0(\mathcal{Z}_a^*,J\mathcal{Z}_a)+i(\mathcal{Z}_a^*,J\mathcal{Z}_e).\end{aligned}\tag{27}$$

The components of the transformed matrix $\hat h_{\rm tr}$ are

$$h_{11}(k)=\begin{cases}-(\mathcal{Z}_a^*,\hat h\mathcal{Z}_e),\ k>0,\\ (\mathcal{Z}_e^*,\hat h\mathcal{Z}_a),\ k<0,\end{cases}$$

$$h_{12}(k)=\begin{cases}(\mathcal{Z}_a^*,\hat h\mathcal{Z}_a),\ k>0,\\ (\mathcal{Z}_e^*,\hat h\mathcal{Z}_e),\ k<0,\end{cases}\tag{28}$$

$$h_{21}(k)=\begin{cases}(\mathcal{Z}_e^*,\hat h\mathcal{Z}_e),\ k>0,\\ (\mathcal{Z}_a^*,\hat h\mathcal{Z}_a),\ k<0,\end{cases}$$

$$h_{22}(k)=\begin{cases}-(\mathcal{Z}_e^*,\hat h\mathcal{Z}_a),\ k>0,\\ (\mathcal{Z}_a^*,\hat h\mathcal{Z}_e),\ k<0.\end{cases}$$

Using equations (19) and (27), we obtain the following expressions for the components of the transformed matrix $\hat h_{\rm tr}$ that define the quadratic Hamiltonian H_2 written in new variables $a(k)$:

$$h_{11}=h_{22}=\omega_0\,\mathrm{sign}\,k,$$

$$h_{12}=-\begin{cases}1,k>0,\\ \delta^2,k<0,\end{cases}$$

$$h_{21}=-\begin{cases}\delta^2,k>0,\\ 1,k>0.\end{cases}$$

Let us introduce the quantity $\tilde\Omega(k)$ such that:

$$h_{21}(k)=h_{12}(-k)=\tilde\Omega(k)=-\begin{cases}\delta^2,k>0\\ 1,k<0\end{cases}$$

Then the expression for H_2 (25) becomes:

$$\begin{aligned}&H_2=\textstyle\int P(k)dk,\\ &P(k)=\frac{\omega_0}{2}\mathrm{sign}k a(k)a(-k)+\mathrm{c.c.}+\tilde\Omega(k)a^*(k)a(k).\end{aligned}\tag{29}$$

Our next step is in taking nonlinearity into account. As was shown by Zakharov (1974), the nonresonant cubic terms H_3 in the expansion of the Hamiltonian H can be eliminated by an appropriate canonical transformation. Assume that this transformation is performed and the third-order terms in H are absent. The fourth-order term describing self-action of a wave-packet has the following form:

$$H_4 = \int W(k_1, k_2, k_3, k_4) a(k_1) a(k_2) a^*(k_3) a^*(k_4) * \\ \delta(k_1 + k_2 - k_3 - k_4) dk_1 dk_2 dk_3 dk_4. \tag{30}$$

To obtain the coefficient $W(k_1, k_2, k_3, k_4)$ one should calculate the third-order as well as the fourth-order terms in the expansion of H. The other terms of the fourth order in our expansion of the Hamiltonian are nonresonant and can be omitted.

So, we present the Hamiltonian H in equation (24) as the sum of the term H_2 of the form (29) and the term H_4 of the form (30). As a result we have the following equations:

$$\begin{aligned} \dot{a}(k) &= -i\omega_0(k)a(k) - i\text{sign } k\tilde{\Omega}(-k)a^*(-k) \\ &\quad -i\text{sign } k\frac{\delta H_4}{\delta a(-k)}, \\ \dot{a}^*(-k) &= -i\omega_0(k)a^*(-k,t) - i\text{sign } k\tilde{\Omega}(k)a(k) \\ &\quad -i\text{sign } k\frac{\delta H_4}{\delta a^*(k)}, \end{aligned} \tag{31}$$

where the last terms describing the self-action of the wave-packet we write down as

$$\frac{\delta H_4}{\delta a^*(k)} = W_1(k_0) \int P(k, k_1, k_2, k_3) dk_1 dk_2 dk_3,$$

$$\frac{\delta H_4}{\delta a(-k)} = W_2(k_0) \int P(k, k_1, k_2, k_3) dk_1 dk_2 dk_3.$$

Here

$$P = a(k_1) a(k_2) a^*(k_3) \delta(k_1 + k_2 - k_3 - k).$$

We omit the other terms in the expressions for $\frac{\delta H_4}{\delta a(-k)}$ and $\frac{\delta H_4}{\delta a(-k)}$, for they are of smaller order with respect to the parameter of nonlinearity, as is shown in what follows. By expanding ω_0 into a series with respect to κ in a vicinity of the point k_0 and leaving the two first terms of the expansion we write $\omega_0 = \hat{\omega}_0 + V\kappa$. Then we extract the central frequency $\hat{\omega}_0$ by introducing the new variable $A(k,t)$ in equations (31):

$$a(k,t) = \exp\,(-i\hat{\omega}_0 t) A(k,t),$$

$$a^*(-k,t) = \exp\,(-i\hat{\omega}_0 t) A^*(-k,t).$$

Now we introduce the new designations $A((k_0+\kappa), t) = A_1(\kappa)$, $A^*(-(k_0+\kappa), t) = A_2(\kappa)$, and assume k_0 to be positive. Then equations (31) can be written as

$$\left(\frac{\partial}{\partial t} + iV\kappa\right) A_1 = iA_2 - iW_2 * \\ \int A_1(k_1) A_1(k_2) A_1^*(k_3) \delta(\kappa + k_1 - k_2 - k_3) dk_1 dk_2 dk_3,$$

$$\left(\frac{\partial}{\partial t} + iV\kappa\right) A_2 = i\delta^2 A_1 - iW_1 * \\ \int A_1(k_1) A_1(k_2) A_1(k_3) \delta(k_1 - k_2 - k_3 - \kappa) dk_1 dk_2 dk_3,$$

where

$$\delta^2 = V_0^2(\kappa^2 - bk_0^2).$$

The inverse Fourier transform of A_1, A_2 with respect to κ leads to the following equations:

$$\begin{aligned} \left(\frac{\partial}{\partial t} + V\frac{\partial}{\partial x}\right) A_1(x) &= iA_2 - 2\pi i W_2 |A_1|^2 A_1, \\ \left(\frac{\partial}{\partial t} + V\frac{\partial}{\partial x}\right) A_2(x) &= -iV_0^2 \left(\frac{\partial^2}{\partial x^2} + b_0 k_0^2\right) A_1 \\ &\quad - 2\pi i W_1 |A_1|^2 A_1. \end{aligned} \tag{32}$$

The small dimensionless parameters in our dynamical system are those of sub or supercriticality b and of nonlinearity ε. In the case of K-H instability the latter is $\varepsilon = \eta k_0$. Let the relation beween these parameters be $b \approx \varepsilon^2$. Then we introduce the suitably scaled variables:

$$A_1 = \varepsilon \tilde{A}_1,\ A_2 = \varepsilon^2 \tilde{A}_2,\ X = \varepsilon x,\ T = \varepsilon t,\ b = \varepsilon^2 b_0.$$

Equations (32) can be rewritten in these variables as

$$\begin{aligned} \left(\frac{\partial}{\partial T} + V\frac{\partial}{\partial X}\right) \tilde{A}_1(X,T) &= i\tilde{A}_2 - 2\pi i W_2 |\tilde{A}_1|^2 \tilde{A}_1, \\ \left(\frac{\partial}{\partial T} + V\frac{\partial}{\partial X}\right) \tilde{A}_2(X,T) &= \\ -iV_0^2 \left(\frac{\partial^2}{\partial X^2} + b_0 k_0^2\right) \tilde{A}_1 &- 2\pi i W_1 |\tilde{A}_1|^2 \tilde{A}_1. \end{aligned} \tag{33}$$

All the terms are of the same order in this system. The last term in the first equation of (32) is omitted in (33) being of higher order in ε. This makes it possible to exclude one of the dependent variables and to obtain the well known nonlinear Klein-Gordon equation:

$$\left(\frac{\partial}{\partial T} + C_1\frac{\partial}{\partial X}\right)\left(\frac{\partial}{\partial T} + C_2\frac{\partial}{\partial X}\right) \tilde{A}_1 - \beta \tilde{A}_1 \\ -2\pi W_1 |\tilde{A}_1|^2 \tilde{A}_1 = 0, \tag{34}$$

where $C_1 = V - V_0$ and $C_2 = V + V_0$, and $\beta = k_0^2 V_0^2 b_0$. Here T and X are slow temporal and spatial coordinates respectively. Equation (34) was obtained by Weissman (1979) for the K-H instability. It arises also in the problem of the buckling of elastic shells. For details see Dodd et al. (1982) and Craik (1985).

Returning to the K-H instability, we obtain the following expressions for the coefficients in the linear part of (34):

$$C_1 = u_1\left(1 - \frac{1}{\sqrt{2}}\right) + u_2\left(1 + \frac{1}{\sqrt{2}}\right),$$

$$C_2 = u_1\left(1 + \frac{1}{\sqrt{2}}\right) + u_2\left(1 - \frac{1}{\sqrt{2}}\right),$$

where $u_1 - u_2 = 2V_c$, and $\beta = 2V_c k_0^2 \Delta V/\varepsilon^2$. To obtain the nonlinear coefficient W_1, the calculation of the Hamiltonians H_3 and H_4 in terms of variables $a(k)$ is necessary.

3 Nonlinear evolution equations for weakly coupled modes

Now we turn ourselves to investigation of another type of weakly unstable media, where the instability is caused by a weak coupling of modes. Our consideration is based on the second example of a geophysical model described in Section 1. This is the three-layer model of an inviscid stably stratified shear flow with continuous velocity profile considered in detail by Goncharov (1986), Romanova (1994), and Romanova (1996).

As was shown by Romanova (1994), the dynamical system in this case can be written in the form (5) with

$$\mathbf{y}(k) = \mathbf{y}^*(-k) = (\Phi_1(k,t),\ \eta_1(k,t),\ \Phi_2(k,t),\ \eta_2(k,t)),$$

where $\eta_j(k,t)$ is the Fourier transform of the j-th disturbed interface, and $\Phi_j(k,t)$ is the Fourier transform of the difference in velocity potentials in two neighbouring layers taken at the j-th interface. The structure of the matrix $J(k)$ is not canonical

$$J(k) = \begin{pmatrix} J_1(k) & 0 \\ 0 & J_2(k) \end{pmatrix},$$

where the second-order matrices $J_j(k)$ are

$$J_j(k) = \begin{pmatrix} 0 & -1 \\ 1 & -i\nu_j/k \end{pmatrix},$$

where $\nu_1 = -V/h, \nu_2 = V/h$. One can easily see that $J(k)$ obeys the constraints defined by equations (6).

Note that the Hamiltonian structure of the system written in these variables is not canonical. In the case of weak nonlinearity the Hamiltonian H can be expanded into a series with respect to η_j and Φ_j, and the first term of this expansion H_2, corresponding to the linear problem, is specified by equation (8), with the matrix $\hat{h}$

$$\hat{h} = \begin{pmatrix} |k|/2 & -iV_1 k & \gamma & 0 \\ iV_1 k & g\lambda_1 + \nu_1 V_1 & 0 & 0 \\ \gamma & 0 & |k|/2 & -iV_2 k \\ 0 & 0 & iV_2 k & g\lambda_2 + \nu_2 V_2 \end{pmatrix} \qquad (35)$$

Here

$\lambda_1 = (\rho_1 - \rho_2)/\rho_1$, $\lambda_2 = (\rho_2 - \rho_3)/\rho_2$,

and $V_1 = -V$, $V_2 = V$. The parameter γ is equal to

$$\gamma = \frac{1}{2}|k| \exp\left(-2|k|h\right).$$

The fourth order dispersion equation specifying the eigenfrequencies ω of the linearised problem is described by equation (9) with the matrix $\hat{h}$ given by equation (35). This leads to the following dispersion equation:

$$\begin{vmatrix} |k|/2 & i\Omega_1 & \gamma & 0 \\ -i\Omega_1 & \dfrac{g\lambda_1 - \nu_1\Omega_1}{k} & 0 & 0 \\ \gamma & 0 & |k|/2 & i\Omega_2 \\ 0 & 0 & -i\Omega_2 & \dfrac{g\lambda_2 - \nu_2\Omega_2}{k} \end{vmatrix} = 0, \qquad (36)$$

where $\Omega_{1,2} = \omega - V_{1,2}k$. The eigenvectors of the linearised system can be determined from the following system of algebraic equations:

$$(\hat{h} - i\omega_j J)\mathbf{z}_j = 0.$$

It is easy to see that dispersion relation (36) can be written in the form:

$$D_1(\omega,k) D_2(\omega,k) = \gamma^2(k), \qquad (37)$$

where

$$D_j(\omega,k) = \left(\frac{|k|}{2} - \frac{\Omega_j}{b_j}\right),$$

and

$$b_j = \frac{g\lambda_j}{\Omega_j} - \frac{\nu_j}{k}. \qquad (38)$$

The equation $D_j(\omega,k) = 0$ is the dispersion relation for the mode related to the j-th interface when the coupling is neglected. It could be checked that if the roots of dispersion equations $D_1 = 0$ and $D_2 = 0$ are well apart, the mutual influence of modes is negligible and the dynamics of wave trains can be considered without taking into account the mode coupling. Now let us see what happens if the dispersion curves related to each interface treated separately intersect at the point (ω_0, k_0), i.e. if $D_1(\omega_0, k_0) = 0$, and $D_2(\omega_0, k_0) = 0$. We consider equation (37) in a vicinity of the point (ω_0, k_0), and let $\omega = \omega_0 + \Delta$, and $k = k_0 + \kappa$, where $\Delta/\omega_0 \ll 1$ and $\kappa/k_0 \ll 1$. Also we substitute $k = k_0$ into the right-hand side of equation (37), and take into account that the dispersion curves cross in the region of large wavenumbers, i.e. $\gamma^2(k_0)/k_0^2 \ll 1$.

Expanding the functions D_1 and D_2 into a series in a vicinity of the point k_0, we obtain the following equation:

$$\Delta^2 - (c_1 + c_2)\kappa\Delta + c_1 c_2 \kappa^2 - \frac{\gamma^2}{D_{1\omega} D_{2\omega}} = 0, \qquad (39)$$

which has the solution:

$$\Delta_{1,2} = \frac{c_1 + c_2}{2}\kappa \pm \sqrt{\left(\frac{c_1 - c_2}{2}\right)^2 \kappa^2 + \frac{\gamma^2}{D_{1\omega} D_{2\omega}}}. \qquad (40)$$

Here

$$c_j = -D'_{jk}(\omega_0, k_0)/D'_{j\omega}(\omega_0, k_0)$$

is the group velocity of the j-th mode treated separately without coupling at the point (ω_0, k_0). One can easily see that the frequencies $\Delta_{1,2}$ take complex values if the quantities $D_{1\omega}$ and $D_{2\omega}$ have different signs. As was shown by many authors (see (Ostrovskiy et al., 1986), etc), the sign of the quantity $D\omega$ specifies the sign of wave energy for the modes described by the equation $D = 0$.

So, instability occurs if the dispersion curves for individual modes crossing at the point (ω_0, k_0) correspond to modes of different energy signs. If the signs of $D_{1\omega}$ and $D_{2\omega}$ are the same, i.e. the wave energy is positive or negative for both waves, the roots of the dispersion equation are real and the waves are stable, but the modes change their identities.

Let us denote

$$s = \frac{\gamma(k_0)}{\sqrt{D_{1\omega}(k_0) D_{2\omega}(k_0)}}. \qquad (41)$$

The dispersion relation of the complete system in a vicinity of the point (ω_0, k_0) can be written as

$$\omega_{1,2} = \omega_0 + \frac{c_1 + c_2}{2}\kappa \pm \sqrt{\left(\frac{c_1 - c_2}{2}\right)^2 \kappa^2 \pm s^2}, \qquad (42)$$

where the minus sign in the radicand corresponds to the unstable coupling of modes, and the plus sign corresponds to the stable coupling which leads to the change of modes identities (See Figure 2).

The approximate dispersion relation in a region of weak instability has the same form as the dispersion relation for waves in the Kelvin-Helmholtz model (see (13)) in the case of weak sub or supercriticality. Nevertheless the evolution equations describing the dynamics of a wave-packet in the spectrally narrow interval of

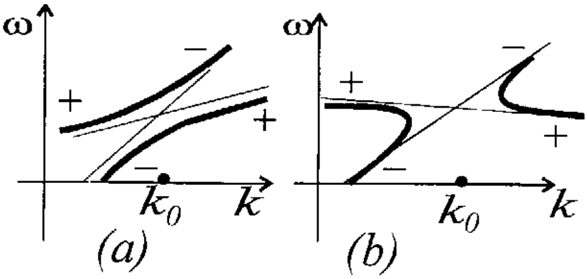

Fig. 2. Figure 2. Dispersion curve for a weak coupling of modes: a)the stable coupling, modes change their identity; b) unstable coupling; the signs are those of wave energy

instability is quite different from the nonlinear Klein-Gordon equation, as we will show below.

It is clear that in the case of weak coupling between modes the normal variables are inappropriate because of the instability, as was shown above, and the variables connected with the notion of the adjoint vector are inappropriate also. To construct the appropriate canonical variables we will introduce the variables connected with the eigenvectors of the linearised system where the weak coupling is ignored. The canonical variables obtained in such a way are not normal ones, i.e. the quadratic part of Hamiltonian H_2 does not have a diagonal form in these variables, but the discreapancy of H_2 from diagonal form is small. Let us denote the matrix $\hat{h}$ defining the Hamiltonian H_2 in the absence of coupling ($\gamma = 0$) as $\hat{h}_0$, and consider the eigenvectors $\mathbf{z}_j$ of the linear problem without accounting for mode connection:

$$(\hat{h}_0 - i\omega_j J)\mathbf{z}_j = 0 \qquad (43)$$

The four linearly independent eigenvectors of the linear algebraic system (43) have the following form:

$$\begin{aligned}\mathbf{z}_{1,2}^{(1)} &= (\phi^{(1)}(\omega_{1,2}),\ \eta^{(1)}(\omega_{1,2}),\ 0,\ 0)\\ &= (c_{1,2}^{(1)},\ ic_{1,2}^{(1)}/b_1(\omega_{1,2}),\ 0,\ 0),\end{aligned}$$

$$\begin{aligned}\mathbf{z}_{1,2}^{(2)} &= (0,\ 0,\ \phi^{(2)}(\omega_{1,2}),\ \eta^{(2)}(\omega_{1,2}))\\ &= (0,\ 0,\ c_{1,2}^{(2)},\ ic_{1,2}^{(2)}/b_2(\omega_{1,2})),\end{aligned}$$

where $c_j^{(p)}$ are arbitrary constants and b_p are defined by equation (38). We introduce the vectors

$$\mathrm{Z}_1(k) = \begin{cases} \mathbf{z}_1^{(1)}, & k > 0, \\ \mathbf{z}_2^{(1)}, & k < 0, \end{cases}$$

$$\mathrm{Z}_2(k) = \begin{cases} \mathbf{z}_1^{(2)}, & k > 0, \\ \mathbf{z}_2^{(2)}, & k < 0. \end{cases}$$

It is easy to show that

$$\mathrm{Z}_1^*(-k) = \begin{cases} \mathbf{z}_2^{(1)}, & k > 0, \\ \mathbf{z}_1^{(1)}, & k < 0, \end{cases}$$

$$Z_2^*(-k) = \begin{cases} \mathbf{z}_2^{(2)}, & k > 0, \\ \mathbf{z}_1^{(2)}, & k < 0. \end{cases}$$

The coordinate form of these vectors is

$$Z_1 = (z_1^{(1)},\ z_2^{(1)},\ 0,\ 0),$$

$$Z_2 = (0,\ 0,\ z_1^{(2)},\ z_2^{(2)}).$$

The transformation from the vector of dependent variables $\mathbf{y}(k)$ to the vector $\mathbf{a}(k)$ connected with the eigenvectors of the uncoupled problem has the following form:

$$\begin{aligned} \mathbf{y}(k) \quad = \quad & Z_1(k)a_1(k) + Z_1^*(-k)a_1^*(-k) \\ & + Z_2(k)a_2(k) + Z_2^*(-k)a_2^*(-k), \end{aligned}$$

or otherwise

$$\mathbf{y}(k) = \mathbf{Z}(k)\mathbf{a}(k),$$

where

$$\mathbf{a}(k) = (a_1(k),\ a_1^*(-k),\ a_2(k),\ a_2^*(-k)).$$

The transformed matrices defining the Poisson structure and the Hamiltonian H_2 have the following form in the new variables $\mathbf{a}(k)$:

$$J_{\text{tr}} = \mathbf{Z}'(-k)J(k)\mathbf{Z}(k),\ \hat{h}_{\text{tr}} = \mathbf{Z}'(-k)\hat{h}(k)\mathbf{Z}(k).$$

The transformed matrix J_{tr} is

$$J_{\text{tr}} = \begin{pmatrix} \mathbf{B}_1 & 0 \\ 0 & \mathbf{B}_2 \end{pmatrix},$$

where components of the matrices $\mathbf{B}_j$ are

$$b_{11}^{(j)} = (Z_j(-k), J_j(k)Z_j(k)),$$

$$b_{12}^{(j)} = (Z_j(-k), J_j(k)Z_j^*(-k)),$$

$$b_{21}^{(j)} = (Z_j^*(k), J_j(k)Z_j(k)),$$

$$b_{22}^{(j)} = (Z_j^*(k), J_j(k)Z_j^*(-k)).$$

Using the properties of eigenvectors corresponding to uncoupled modes, we can show just as it has been done by Romanova (1994) that

$$(\mathbf{z}_j^{(p)*}(k), J^{(p)}(k)\mathbf{z}_l^{(q)}(k)) = -i\frac{\partial D_p}{\partial \omega_j} c_j^{(p)} c_j^{(p)*} \delta_q^p \delta_l^j,$$

where δ_q^p is the Kronecker delta, and

$$\frac{\partial D_p}{\partial \omega_j} = \frac{2b_j^{(p)} + \nu^{(p)}/k}{b_j^{(p)2}}.$$

If we assume the arbitrary constants to be

$$c_j^{(p)} = \left(\frac{\partial D_p}{\partial \omega_j}\right)^{-\frac{1}{2}}, \tag{44}$$

the matrix J_{tr} takes the canonical form, and

$$\mathbf{B}_1 = \mathbf{B}_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}. \tag{45}$$

It follows from (45) and (5) that the dynamical system written in the variables $\mathbf{a}(k)$ has the standard form

$$\dot{a}_j(k,t) = -i\frac{\delta H}{\delta a_j^*(k)} \tag{46}$$

Dynamical equation (46) is quite different from equation (24) which describes the dynamics of weakly unstable wave-packets in the K-H model. This equation was obtained by Zakharov (1968) to describe the dynamics of the stable waves in normal variables. However the matrix $\hat{h}_{\text{tr}}$ which defines the quadratic Hamiltonian H_2 written in new variables does not possess the normal form, for the eigenvectors $\mathbf{z}_l^{(q)}$ are not normal with respect to our complete system, where the small parameter of mode connection γ is taken into account. Omitting calculation, we write down the matrix $\hat{h}_{\text{tr}}$ in the following form:

$$\hat{h}_{\text{tr}} = \begin{pmatrix} \hat{h}^{(1)} & \gamma\mathbf{F}_1 \\ \gamma\mathbf{F}_2 & \hat{h}^{(2)} \end{pmatrix},$$

where the matrices $\mathbf{F}_1$ and $\mathbf{F}_2$ are

$$\mathbf{F}_1(k) = \begin{pmatrix} z_1^{(1)}(-k)z_1^{(2)}(k) & z_1^{(1)}(-k)z_1^{(2)*}(-k) \\ z_1^{(1)*}(k)z_1^{(2)}(-k) & z_1^{(1)*}(k)z_1^{(2)*}(-k) \end{pmatrix},$$

$$\mathbf{F}_2(k) = \mathbf{F}_1'(-k).$$

Using equation (44), which leads to

$$z_1^{(j)}(k) = \frac{1}{\sqrt{D'_{j\omega}(k)}},$$

we obtain

$$\mathbf{F}_1(k) = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} = \begin{pmatrix} \sqrt{D'_{1\omega}(-k)D'_{2\omega}(k)} & \sqrt{D'_{1\omega}(-k)D'_{2\omega}(-k)} \\ \sqrt{D'_{1\omega}(k)D'_{2\omega}(k)} & \sqrt{D'_{1\omega}(k)D'_{2\omega}(-k)} \end{pmatrix}. \tag{47}$$

Note that all the terms in the matrix $\mathbf{F}_1(k)$ are real, for we always choose those branches of the dispersion curve for which signs of adiabatic invariants are positive, i.e. $D\omega'(k) > 0$. The matrices $\hat{h}^{(j)}$ have the following components

$$\begin{aligned}
h_{11}^{(j)} &= (\mathrm{Z}_j(-k), \hat{h}^{(j)}(k)\mathrm{Z}_j(k)) = 0, \\
h_{12}^{(j)} &= (\mathrm{Z}_j(-k), \hat{h}^{(j)}(k)\mathrm{Z}_j^*(-k)) \\
&= \begin{cases} i\omega_2^{(j)}(\mathbf{z}_2^{(j)}, J^{(j)}\mathbf{z}_2^{(j)}),\ k>0 \\ i\omega_1^{(j)}(\mathbf{z}_1^{(j)}, J^{(j)}\mathbf{z}_1^{(j)}),\ k<0 \end{cases} \\
&= \begin{cases} -\omega_2^{(j)},\ k>0 \\ -\omega_1^{(j)},\ k<0 \end{cases} = \Omega^{(j)}(-k), \\
h_{21}^{(j)} &= (\mathrm{Z}_j^*(k), \hat{h}^{(j)}(k)\mathrm{Z}_j(k)) \\
&= \begin{cases} i\omega_1^{(j)}(\mathbf{z}_1^{(j)}, J^{(j)}\mathbf{z}_1^{(j)}),\ k>0 \\ i\omega_2^{(j)}(\mathbf{z}_2^{(j)}, J^{(j)}\mathbf{z}_2^{(j)}),\ k<0 \end{cases} \\
&= \begin{cases} \omega_1(j),\ k>0 \\ \omega_2(j),\ k<0 \end{cases} = \Omega^{(j)}(k), \\
h_{22}^{(j)} &= (\mathrm{Z}_j^*(k), \hat{h}^{(j)}(k)\mathrm{Z}_j^*(-k)) = 0.
\end{aligned} \tag{48}$$

As a result the quadratic part of Hamiltonian H_2 has the form

$$\begin{aligned}
H_2 = \textstyle\int (\sum_{j=1}^{2} \Omega^{(j)}(k) a_j(k) a_j^*(k) + \\
(f_{11}(k)a_2(k)a_1(-k) + f_{21}(k)a_2(k)a_1^*(k) + \text{c.c.}))dk
\end{aligned} \tag{49}$$

On substituting the Hamiltonian H in the form

$$H = H_2 + \varepsilon^2 H_4 + \ldots,$$

where H_2 is defined by equation (49), into equation (46), we obtain the following system:

$$\begin{aligned}
\dot{a}_1(k) &= -i\Omega_1(k)a_1(k) - i\frac{\delta H_4}{\delta a_1^*(k)} \\
&\quad -i\gamma[f_{21}(k)a_2(k) + f_{22}(k)a_2^*(-k)], \\
\dot{a}_2(k) &= -i\Omega_2(k)a_1(k) - i\frac{\delta H_4}{\delta a_2^*(k)} \\
&\quad -i\gamma[f_{21}(k)a_1(k) + f_{11}(k)a_1^*(-k)],
\end{aligned} \tag{50}$$

and

$$\begin{aligned}
\dot{a}_1^*(-k) &= i\Omega_1(-k)a_1^*(-k) + i\frac{\delta H_4}{\delta a_1(-k)} \\
&\quad +i\gamma[f_{11}(k)a_2(k) + f_{12}(k)a_2^*(-k)], \\
\dot{a}_2^*(-k) &= i\Omega_2(-k)a_2^*(-k) + i\frac{\delta H_4}{\delta a_2(-k)} \\
&\quad +i\gamma[f_{22}(k)a_1(k) + f_{12}(k)a_1^*(-k)].
\end{aligned} \tag{51}$$

The cubic term H_3 in the expansion of the Hamiltonian is omitted for we assume that there are no resonant terms of this order. We also assume the canonical transformation excluding nonresonant terms to have been performed. But the resonant term H_4 of the fourth order responsible for the self-action of a nonlinear wave is always present. The transformation excluding the nonresonant term H_3 does not change the structure of the resonant term H_4 but influences the interaction coefficients. We consider the case when the coupling of different modes is weak, and as a consequence the terms with coefficients proportional to the small parameter of connection γ can be neglected in the nonlinear terms. It follows then that H_4 can be written in the form:

$$\begin{aligned}
&H_4 = \tfrac{1}{2}\textstyle\int (W_1(k_1,k_2,k_3,k_4)a_1(k_1)a_1(k_2)a_1^*(k_3)a_1^*(k_4) \\
&+W_2(k_1,k_2,k_3,k_4)a_2(k_1)a_2(k_2)a_2^*(k_3)a_2^*(k_4))* \\
&\delta(k_1+k_2-k_3-k_4)dk_1dk_2dk_3dk_4
\end{aligned}$$

Apparently, if we take a wave-packet centered on a certain point where dispersion curves of the linearised system are far apart, we can neglect the terms proportional to γ for their consideration does not contribute to the slow modulation of a wave-packet in this case. As a result we obtain the standard nonlinear Schrödinger equation from equations (50) and (51) for each mode. The matter changes if we consider the dynamics of a wave-packet from the interval of wave-numbers where the linear resonance between modes takes place. There are two types of such a resonance. The first one corresponds to the case when the eigenfrequencies of the two modes considered separately coincide at a point k_0: $\Omega_1(k_0) = \Omega_2(k_0) = \Omega_0$. The second type of a resonance takes place when the equality $\Omega_1(k_0) = -\Omega_2(-k_0) = \Omega_0$ holds.

Let us consider the first case. We introduce the new variables $A_j(k,t)$ by extracting the central frequency Ω_0:

$$a_j(k,t) = \exp(-i\Omega_0 t)A_j(k,t), \tag{52}$$

and write down the equations for $A_j(k,t)$ which follow from equations (50, 51):

$$\begin{aligned}
\dot{A}_1(k) &= -i\Omega'_{1k}(k_0)\kappa A_1(k) - i\frac{\delta H_4}{\delta A_1^*(k)} \\
&\quad -i\gamma f_{21}(k_0)A_2(k), \\
\dot{A}_2(k) &= = -i\Omega'_{2k}(k_0)\kappa A_2(k) - i\frac{\delta H_4}{\delta A_2^*(k)} \\
&\quad -i\gamma f_{21}(k_0)A_1(k).
\end{aligned} \tag{53}$$

Here we neglected small linear terms that are not resonant and all the terms but linear in the expansion of the functions $\Omega_j(k)$ in a vicinity of the intersection point k_0 with respect to $\kappa = k - k_0$. We denote the quantities $\Omega'_{jk}(k)$ as $c_{j\mathrm{gr}}$. They have the sense of the group velocities at the point k_0 for each mode considered separately.

Now let us turn to the coefficient $f_{21}(k_0)$. As was shown by Romanova (1994), the method we have used for the construction of our canonical variables demands the expressions $D'_{1\omega}(k_0)$ and $D'_{2\omega}(k_0)$ to be positive. Recall that these quantities have the meaning of adiabatic invariants for each mode considered separately. It follows from (47) that

$$f_{21}(k_0) = \frac{1}{\sqrt{D'_{1\omega}(k_0)}}\frac{1}{\sqrt{D'_{2\omega}(k_0)}}$$

Let us introduce the quantity $s = \gamma f_{21}(k_0)$. On substituting the expression for H_4 into equations (53), we obtain the following system of equations:

$$\dot{A}_1(k,t) = -ic_{1\mathrm{gr}}\kappa A_1(k,t) - isA_2(k,t)$$

$$- iW_1 \int R_1(k_1,k_2,k_3,\kappa)dk_1dk_2dk_3,$$

$$\dot{A}_2(k,t) = -ic_{2\mathrm{gr}}\kappa A_2(k,t) - isA_1(k,t)$$

$$- iW_2 \int R_2(k_1,k_2,k_3,\kappa)dk_1dk_2dk_3,$$

where

$$R_1 = A_1(k_1)A_1(k_2)A_1^*(k_3)\delta(k_1+k_2-k_3-\kappa),$$

$$R_2 = A_2(k_1)A_2(k_2)A_2^*(k_3)\delta(k_1+k_2-k_3-\kappa),$$

and

$$W_{1,2} = W_{1,2}(k_0,k_0,k_0,k_0).$$

Now we assume the small parameter s defining the weak coupling of modes to be of the order of ε^2 and consider the "slow" time $T = t\varepsilon^2$ and the "slow" coordinate $X = x\varepsilon^2$, where ε is the small parameter of nonlinearity. We perform the inverse Fourier transform with respect to the variable $K = \kappa/\varepsilon^2$. Then we introduce the parameter $S = s/\varepsilon^2$ and as a result we obtain the final system, all the terms of which have the same order:

$$\begin{aligned}\frac{\partial A_1}{\partial T} + c_{1\mathrm{gr}}\frac{\partial A_1}{\partial X} - iSA_2 - 2\pi i W_1|A_1|^2A_1 = 0,\\ \frac{\partial A_2}{\partial \tau} + c_{2\mathrm{gr}}\frac{\partial A_2}{\partial X} - iSA_1 - 2\pi i W_2|A_2|^2A_2 = 0.\end{aligned} \tag{54}$$

As can be easily seen, the dispersion equation related to the linearised system (54) has the form (42) with the sign "plus" in the radicand. This case corresponds to the stable situation when the coupled modes interchange their identities.

In quite a similar manner we obtain evolution equations in a vicinity of a point k_0 in the second case, when the equation $\Omega_1(k_0) = -\Omega_2(-k_0) = \Omega_0$ holds. Then the coupling of modes causes instability, as will be shown later. In this case it follows from equations (50) and (51) that (the nonresonant linear terms are omitted):

$$\begin{aligned}\dot{a}_1(k) &= -i\Omega_1(k)a_1(k) - i\frac{\delta H_4}{\delta a_1^*(k)}\\ &\quad -i\gamma f_{22}(k_0)a_2^*(-k),\\ \dot{a}_2^*(-k) &= -i(-\Omega_2(-k))a_2^*(-k) + i\frac{\delta H_4}{\delta a_2(-k)}\\ &\quad +i\gamma f_{22}(k_0)a_1(k).\end{aligned}$$

The following procedure for deriving the evolution equation is quite similar to that performed above for the stable case. By analogy with the transformation (52), we extract the central frequency Ω_0 and introduce new variables $A(k,t)$, $B(k,t)$ having the sense of slowly varying amplitudes of our wave-packets:

$$a_1(k,t) = \exp(-i\Omega_0 t)A(k,t),$$

$$a_2^*(-k,t) = \exp(-i\Omega_0 t)B(k,t).$$

As a result we obtain the following evolution system:

$$\begin{aligned}\frac{\partial A}{\partial T} + c_{1gr}\frac{\partial A}{\partial X} - iSB - 2\pi i W_1|A|^2A = 0,\\ \frac{\partial B}{\partial T} + c_{2gr}\frac{\partial B}{\partial X} + iSA + 2\pi i W_2|B|^2B = 0.\end{aligned} \tag{55}$$

Here we introduced the quantity $S = \gamma f_{22}(k_0)/\varepsilon^2$ where the coefficient f_{22} is

$$f_{22}(k_0) = \frac{1}{\sqrt{D'_{1\omega}(k_0)}}\frac{1}{\sqrt{D'_{2\omega}(-k_0)}}.$$

The group velocities of uncoupled modes at the point k_0 are

$$c_{1\mathrm{gr}} = \Omega'_{1k}(-k_0),\ c_{2\mathrm{gr}} = -\Omega'_{2k}(-k_0).$$

So, the evolution equations (55) describe the dynamics of wave-packets in the region of unstable coupling between modes, when the uncoupled modes which cross at the point k_0 have different energy signs.

4 Algebraic instability in the framework of a single mode

As we can have demonstrated, the evolution equations have quite a different form in the case of the Kelvin-Helmholtz instability and in the case of the weak coupling of modes. The reason is that the intrinsic structure of dynamical equations in a region of weak instability is quite different for these two cases. In both of them we have a small parameter, in the first case it is a deviation of a velocity jump in a shear flow from its critical value (parameter b), and in the second case it is the parameter s defining the weak coupling of modes. In both cases the two eigenfrequencies coincide at the central point of a wave-packet if the parameters are equal to zero, but the structure of eigenvectors turns out to be quite different. In the first case we have the eigenvector and the adjoint eigenvector corresponding to the multiple value of the eigenfrequency, and in the second one two independent eigenvectors belonging to different subspaces correspond to this eigenfrequency, i.e. the intersecting modes become uncoupled and independent. In the first case (the K-H instability) two waves remain coupled if $b = 0$, i.e. when the parameter of supercriticality is equal to zero, and they cannot be considered separately. That is why we can speak about instability in the framework of a single mode. From mathematical viewpoint, we cannot find the appropriate transformation reducing the matrix $\hat{h}$ which specifies the quadratic part of the Hamiltonian, to the diagonal form. It has the form of the Jordan box at the point k_0 and the form close to it in a vicinity of the wavenumber k_0. Any other transformation will lead to a bad normalization and will destroy the approximation of weak nonlinearity.

So, if we consider the instability in the framework of a single mode, as is the case with the Kelvin-Helmholtz instability, we perform the transformation connected with the eigenvector and the adjoint eigenvector. This transformation results in the canonical structure in the form of (24). Note that if the matrix describing the Hamiltonian in the linearised system has the form of the Jordan box at the point k_0, the result is an algebraic instability, when one of dependent variables grows linearly with time. It leads to the different order of variables with respect to the small nonlinearity parameter when we consider the nonlinear problems. But if we consider the instability due to the weak coupling of modes, the situation is quite different. If the parameter of coupling is equal to zero, the matrix defining the Hamiltonian of the linearised system is diagonal with equal eigenfrequencies. There is no instability in this case.

Now we turn to an algebraic instability. Let us consider the linearised Kelvin-Helmholtz weakly subcritical problem problem in the stable case for a single harmonic component. Then the equations follow from (32):

$$\dot{A}_1(k,t) = iA_2(k,t),$$
$$\dot{A}_2(k,t) = is^2 A_1(k,t). \tag{56}$$

The linearised problem for the waves in the region of a stable coupling of modes is

$$\dot{A}_1(k,t) = -isA_2(k,t),$$
$$\dot{A}_2(k,t) = -isA_1(k,t). \tag{57}$$

Both the cases give us the same equation:

$$\ddot{A}_1 + s_1^2 A_1 = 0.$$

The solution of system (56), subject to the initial data $A_1(0) = A_{10},\ A_2(0) = A_{20}$, has the following form:

$$A_1(t) = A_{10}\cos(st) - \tfrac{1}{s}A_{20}\sin(st),$$
$$A_2(t) = sA_{10}\sin(st) + A_{20}\cos(st). \tag{58}$$

As we can see, the quantity $A_1(t)$ grows up to a definite value having the order of s^{-1}, and the second variable $A_2(t)$ does not grow with time. The first variable $A_1(t)$ grows until the fraction $A_2(t)/A_1(t)$ has the order s, then it stops growing. So, at the initial stage the solution grows linearly with time up to a definite value, and if s tends to zero, we obtain the linear growth up to infinity. In the case of the stable wave coupling the solution of system (57) has the form

$$A_1(t) = A_{10}\cos(st) - A_{20}\sin(st),$$
$$A_2(t) = -A_{10}\sin(st) - A_{20}\cos(st). \tag{59}$$

No growth is observed for the case of stable coupling of modes. There is no threshold instability in this case.

5 Discussions

In this paper we have outlined the general approach to the derivation of evolution equations for weakly unstable wave-packets propagating in unstable media. Three types of weakly unstable wave-packets could be distinguished. The first type is a wave-packet centered at the point of marginal stability, when the instability in the medium is strongly supercritical, and the neighbouring interval of linear instability is large compared to the spectral width of the wave-packet. In the general case the evolution equation for such wave-packets was obtained by Romanova (1994). For the Kelvin-Helholtz type of instability this equation was derived by several authors (see Craik (1985)). This equation has the same form as the well known nonlinear Schrödinger equation, but with the roles of X and T interchanged:

$$\left(\frac{\partial}{\partial T} - c(k_0)\frac{\partial}{\partial X}\right)^2 A - ib(k_0)\frac{\partial A}{\partial X} = iT|A|^2 A.$$

The second case which was considered in this paper is the case of a weak intramodal instability of Kelvin-Helmholtz type. The purpose of our investigations was the derivation of evolution equations governing the nonlinear dynamics of wave-packets enclosing the spectrally narrow interval of unstable waves. This derivation was performed based on the introduction of canonical variables valid for the region of a weak instability in the framework of a single mode. As a result we obtained the well known nonlinear Klein-Gordon equation.

The third case also considered is : weak instability due to weak coupling of modes having different energy signs. The evolution equations have a quite different form in this case. This is the system of the two coupled nonlinear Schrödinger equations. We cannot exclude one of the dependent variables for they have the same order with respect to the parameter of nonlinearity. The suggestion made by Romanova (1994) that the unstable waves in the region of weak coupling of modes are described by the nonlinear Klein-Gordon equation is erroneous.

The purpose of our further investigations is the derivation of evolution equations for three-wave resonant interactions involving the weakly unstable mode. The case when this mode in marginally unstable is considered by Romanova (1998). In this case the three wave resonant amplitude equations have the form:

$$\begin{aligned} A_{1\tau} &= A_2^* A_3^*, \\ A_{2\tau} &= A_1^* A_3^*, \\ A_{3\tau\tau} &= -iA_1^* A_2^* - T|A_3|^2 A_3, \end{aligned}$$

where $A_{1,2}$ are the amplitudes of stable waves, and A_3 is the amplitude of a marginally stable wave.

The amplitude equations in two other cases is the subject for further investigation.

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